Partial Solution Set, Leon §6.5

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6.5.1 We are to show first of all that A and A^T have the same nonzero singular values, then to describe the relationship between the singular value decompositions of A and A^T. So begin by assuming that σ is a nonzero singular value for A. By definition of singular value, we know that $\lambda = \sigma^2$ is a positive eigenvalue of A^TA. Let $\mathbf{x}$ be an eigenvector for A^TA belonging to λ. Then $A^TA\mathbf{x} = \lambda\mathbf{x}$. So

$$\begin{aligned}\lambda(A\mathbf{x}) &= A(\lambda\mathbf{x})\\ &= A(A^TA\mathbf{x})\\ &= AA^T(A\mathbf{x}),\end{aligned}$$

so $A\mathbf{x}$ is an eigenvector for AA^T, also belonging to λ. Conversely, suppose that σ is a nonzero singular value for A^T. Then $\lambda = \sigma^2$ is a positive eigenvalue for AA^T, with eigenvector $\mathbf{x}$, i.e., $AA^T\mathbf{x} = \lambda\mathbf{x}$. Then

$$\begin{aligned}\lambda(A^T\mathbf{x}) &= A^T(\lambda\mathbf{x})\\ &= A^T(AA^T\mathbf{x})\\ &= A^TA(A^T\mathbf{x}),\end{aligned}$$

so $A^T\mathbf{x}$ is an eigenvector for A^TA, also belonging to λ. Thus AA^T and A^TA have the same positive eigenvalues, hence the same nonzero singular values. □

How, then, are the singular value decompositions for A and A^T related? This is more easily answered: if $A = U\Sigma V^T$, then $A^T = (U\Sigma V^T)^T = V\Sigma^T U^T$.

6.5.2 We are to find the singular value decompositions of several matrices, using the method outlined in the text. Here are two of the solutions.

(a) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. We begin by finding $A^TA = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 10$ and $\lambda_2 = 0$. So $\sigma_1 = \sqrt{10}$, and $\sigma_2 = 0$, and we have $\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}$. We know that V is a diagonalizing matrix for A^TA, and that we want the first column of V to be a unit eigenvector for λ_1. We choose $\mathbf{v}_1 = (1/\sqrt{2}, 1/\sqrt{2})^T$. The second eigenvector must belong to λ_2; we choose $\mathbf{v}_2 = (1/\sqrt{2}, -1/\sqrt{2})^T$. So $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$. Now we must find U. The first column of U is obtained from the equation

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

For the second column of U, we find a unit vector from $N(A^T)$; we take $\mathbf{u}_2 = (2/\sqrt{5}, -1/\sqrt{5})^T$. The singular value decomposition of A is therefore

$$A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

(c) We have $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Proceding as in the previous problem, we first find $A^TA = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$. The eigenvalues of A^TA are $\lambda_1 = 16$ and $\lambda_2 = 4$, with associated eigenvectors $\mathbf{v}_1 = (1/\sqrt{2}, 1/\sqrt{2})^T$ and $\mathbf{v}_2 = (1/\sqrt{2}, -1/\sqrt{2})^T$. The singular values are $\sigma_1 = 4$, and $\sigma_2 = 2$, and $\Sigma = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. We find $\mathbf{u}_1 = \frac{1}{4}A\mathbf{v}_1 = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0)^T$, and $\mathbf{u}_2 = \frac{1}{2}A\mathbf{v}_2 = (-1/\sqrt{2}, 1/\sqrt{2}, 0, 0)^T$. It remains to find a pair of orthogonal unit vectors from $N(A^T)$; $\mathbf{u}_3 = (0, 0, 1, 0)^T$ and $\mathbf{u}_4 = (0, 0, 0, 1)^T$ will do nicely. Finally, we have

$$A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

Note that we can also obtain a more compact factorization of A by discarding columns three of U and rows three and four of Σ, obtaining a more compact factorization $A = U_1\Sigma_1 V^T$.

6.5.3 For the matrices in problem 6.5.2 whose SVDs were found above, the first has rank 1, while the second has rank 2. Since $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ already has rank one, it is its own best rank-one approximation (and a very good approximation it is). The closest rank-one approximation to $B = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is obtained by replacing its least singular value $\sigma_2 = 2$ with 0; the resulting factorization give us $\hat{B} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

6.5.4 We have $A = \begin{bmatrix} -2 & 8 & 20 \\ 14 & 19 & 10 \\ 2 & -2 & 1 \end{bmatrix}$, with singular value decomposition

$$A = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

The closest rank-two matrix to A is $B = \begin{bmatrix} -2 & 8 & 20 \\ 14 & 19 & 10 \\ 0 & 0 & 0 \end{bmatrix}$, and the closest rank-one matrix to A is $C = \begin{bmatrix} 6 & 12 & 12 \\ 8 & 16 & 16 \\ 0 & 0 & 0 \end{bmatrix}$.

6.5.5 The matrix $A = \begin{bmatrix} 2 & 5 & 4 \\ 6 & 3 & 0 \\ 6 & 3 & 0 \\ 2 & 5 & 4 \end{bmatrix}$ has singular value decomposition

$$A = U\Sigma V^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

(a) Use the singular value decomposition to find orthonormal bases for $R(A^T)$ and $N(A)$.

Solution: By inspecting Σ, we see that $\text{rank}(A) = 2$. It follows that the first two columns of V (rows of V^T) are an orthonormal basis for $R(A^T)$, while the last column of V is an orthonormal basis for $N(A)$.

(b) As above, but for $R(A)$ and $N(A^T)$.

Solution: The first two columns of U are an orthonormal basis for $R(A)$, and the third and fourth columns are an orthonormal basis for $N(A^T)$.

6.5.9 Let A be a matrix of rank n with SVD $U\Sigma V^T$. Let Σ^+ $n \times m$ matrix shown:

$$\Sigma^+ = \left[\begin{array}{cccc|c} \frac{1}{\sigma_1} & & & & \\ & \frac{1}{\sigma_2} & & & 0 \\ & & \ddots & & \\ & & & \frac{1}{\sigma_n} & \end{array}\right].$$

Define A^+ by $A^+ = V\Sigma^+U^T$. Show that $\hat{\mathbf{x}} = A^+\mathbf{b}$ satisfies the normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$.

Proof: Let $\mathbf{b} \in \mathbf{R}^m$, and let A, A^+, and $\hat{\mathbf{x}}$ be as described. Then

$$\begin{aligned}
A^TA\hat{\mathbf{x}} &= AA^TAA^+\mathbf{b} \\
&= V\Sigma^TU^TU\Sigma V^TV\Sigma^+U^T\mathbf{b} \\
&= V\Sigma^TU^TU\Sigma\Sigma^+U^T\mathbf{b} && \text{(Since } V^TV = I\text{)} \\
&= V\Sigma^TU^TUU^T\mathbf{b} && \text{(Since } \Sigma\Sigma^+ = I\text{)} \\
&= V\Sigma^TU^T\mathbf{b} && \text{(Since } U^TUU^T = U^T\text{)} \\
&= A^T\mathbf{b},
\end{aligned}$$

and we're done. $\square$

MA/Ra, November 27, 2002